Problem 1.

A bead of mass $m$ slides without friction on a wire in the $(xy)$ plane with the shape of a cycloid, with coordinates

\[ x = a(\theta - \sin \theta) \]
\[ y = a(1 + \cos \theta) \]

where $0 \leq \theta \leq 2\pi$. Find the Lagrangian when the setup is placed in a uniform gravitational field of strength $g$ pointing in the negative $y$-direction. Find the equation of motion. Show that in the coordinate $s(t) = \cos(\theta(t)/2)$ the equation of motion is

\[ \frac{d^2s(t)}{dt^2} + \omega^2 s(t) = 0 \]

What is $\omega$? Integrate the equation of motion and find how long it takes for the bead to go from $x = 0$ to the point where $y = 0$, for $a = 10 \text{ cm}$, $m = 2 \text{ kg}$. Assume the initial velocity is zero.
Solution: Spring 1997

Probl. 1

\[ L = m \frac{x^2 + y^2}{2} - mg y \]

\[ x = a (\theta - \sin \theta) \]
\[ y = a (1 + \cos \theta) \]

\[ \dot{x} = a \left( \ddot{\theta} - \dot{\theta} \cos \theta \right) = a \dot{\theta} (1 - \cos \theta) \]
\[ \dot{y} = -a \dot{\theta} \sin \theta \]

\[ L = \frac{m a^2 \dot{\theta}^2}{2} \left\{ (1 - \cos \theta)^2 + \sin^2 \theta \right\} - mga (1 + \cos \theta) \]

\[ = ma^2 \dot{\theta}^2 (1 - \cos \theta) - mga (1 + \cos \theta) \]

\[ s = \cos \frac{\theta}{2} \]
\[ S = \cos \frac{\theta}{2} \]

\[ \ddot{S} = -\frac{1}{2} \sin \frac{\theta}{2} \dot{\theta} \]
\[ S^2 = \frac{\dot{\theta} \sin^2 \theta}{4} = \frac{1}{8} (1 - \cos \theta) \dot{\theta}^2 \]

\[ S^2 = \cos^2 \frac{\theta}{2} = 1 - \frac{1}{2} (1 - \cos \theta) = \frac{1 + \cos \theta}{2} \]

\[ L = 8ma^2 \dot{s}^2 - 2mga s^2 \]

\[ \frac{\partial L}{\partial \dot{s}} = 8ma^2 \dot{s} \]
\[ \frac{\partial L}{\partial s} = -4mga s \]
\[16ma^2 \ddot{s} + 4mgas = 0\]
\[\ddot{s} + \frac{g}{4a} s = 0\]

\[\omega = \sqrt{\frac{g}{4a}}\]

\[s(t) = cn\sqrt{\frac{g}{4a}} t = cn \frac{\theta(t)}{2}\]

\[\theta(t) = \sqrt{\frac{g}{a}} t\]

For \(\theta = \pi, \ y = 0, \ t(y=0) = \pi \sqrt{\frac{a}{g}}\]
Problem 2.

A globular cluster is a gravitationally bound association of stars.

Derive a rough estimate of the relaxation time through two body interactions, that is, the time for the average star to experience one strong interaction which significantly changes its velocity $v$. Assume that all stars have the same mass $m$, and that the average number density of stars is $n$ (stars/volume). Your answer should not depend on the size of the stars, which can be neglected. Give a numerical estimate for the relaxation time for a cluster of stars of typical mass $m = 2 \times 10^{33}$ g, typical velocity $v = 5$ km/s, and density $n = 10^6$ stars per pc$^3$ (pc = $3 \times 10^{18}$ cm).

Data: $G = 6.67 \times 10^{-8}$ gm$^{-1}$ cm$^3$s$^{-2}$
Solution: Spring 1997
Prob. 2

Relaxation Time parts 1(b), 1(c)

a) Let each star have "sphere of influence" with cross-sectional area \( \pi r^2 \), where \( r \) defined below.

Strong interactions occur when a second star passes inside \( r \).

Define \( t_r = \text{relaxation time} = \text{time between strong interactions} \).

Cylindrical volume swept out by one star in time \( t_r \) is

\[ \pi r^2 v t_r \]

where \( v \) = velocity.

By definition of \( t_r \), \( \pi r^2 v t_r = \frac{t}{n} \) stars/volume

\[ \Rightarrow t_r = \frac{1}{\pi r^2 vn} \]

Choose \( r \) such that grav. potential energy of a pair of stars = typical random kinetic energy:

\[ Gm^2/r = m v^2/2 \]

\[ \Rightarrow r = \frac{2GM}{v^2} \]

Then \( t_r = \frac{1}{\pi r^2 vn} = \frac{1}{\pi (2GM/v^2)^2 VN} = \frac{V^3}{4\pi G^2 m^2 n} \)
Problem 3.

A particle of mass \( m \) in a one-dimensional box extending from \( x = 0 \) to \( x = L \) has wavefunction \( \psi(x) = C \), \( L/3 \leq x \leq 2L/3 \), and zero otherwise.

a) Determine the normalization constant \( C \) of the wavefunction.

b) Find the average value of \( x \), the position of the particle.

c) Find the probability to measure the particle in the ground state and the first two excited states.

d) Use the results of part (c) to write the wavefunction in terms of the energy eigenstates, \( \psi(x) = c_n \psi_n \). Check your answer by verifying that the probabilities for the particle to be in the different energy eigenstates sum to one, i.e. \( \sum_{n=1,2,3,\ldots} |c_n|^2 = 1 \).

The identities

\[
\sum_{n=2,4,\ldots} \frac{1}{n^2} = \frac{\pi^2}{24},
\]

\[
\sum_{n=1,3,\ldots} \frac{1}{n^2} = \frac{\pi^2}{8},
\]

may be helpful.

e) What is the expectation value for the energy of the particle?
Solution: Spring 1997

Prob. 3

1. \[ \psi(x) \]

\[ \int \psi(x) \psi^*(x) \, dx = 1 = C^2 \int_{V_3}^2 \, dx = C^2 \frac{L}{3}. \]

\[ \Rightarrow \quad C = \sqrt{\frac{3}{L}} \]

2. \[ \langle \psi \mid x \mid \psi \rangle = \int_{V_3}^2 \psi^*(x) \psi(x) \, dx = C^2 \frac{x^2}{2} \bigg|_{V_3}^{2} \]

\[ = \frac{C^2 L^2}{6} = \frac{L}{2} \]

3. \[ \psi_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3 \ldots \]

\[ \int_{V_3}^2 \psi^*(x) \psi(x) \, dx = \sqrt{\frac{2}{L}} \, C \int_{V_3}^2 \sin \frac{n\pi x}{L} \, dx \]

\[ = -\frac{CL}{\pi} \sqrt{\frac{2}{L}} \cos \frac{n\pi x}{L} \bigg|_{V_3}^{2} \]

\[ = -\frac{CL}{\pi} \sqrt{\frac{2}{L}} \left( \cos \frac{2\pi}{3} - \cos \frac{\pi}{3} \right) = \frac{CL}{\pi} \sqrt{\frac{2}{L}} \frac{2L}{2} \]

\[ = \frac{\sqrt{6}}{\pi} \]

\( n=2 \) \[
\int_{\frac{1}{3}}^{\frac{2}{3}} \psi^*(x) \psi_2(x) \, dx = 0 \text{ since } \psi(x) \text{ is even.}
\]

\( n=3 \) \[
\int_{\frac{1}{3}}^{\frac{2}{3}} \psi^*(x) \psi_3(x) \, dx = \sqrt{\frac{2}{l}} \, C \int \sin \frac{3\pi x}{l} \, dx
\]
\[= -\frac{CL}{3\pi} \sqrt{\frac{2}{L}} \left( \cos \frac{6\pi}{3} - \cos \frac{3\pi}{3} \right) \]
\[= -\frac{2\sqrt{6}}{3\pi} \]

\( \psi(x) = \sum_n c_n \psi_n \)

\( c_n = 0 \text{ if } n \text{ even} \)

\( c_1 = \frac{\sqrt{6}}{\pi} \)
\( c_5 = \frac{1}{5} \left( \frac{\sqrt{6}}{\pi} \right) \)
\( c_7 = \frac{1}{7} \left( \frac{\sqrt{6}}{\pi} \right) \)
\( c_9 = \frac{1}{9} \left( \frac{\sqrt{6}}{\pi} \right) \)
\( c_{11} = \frac{1}{11} \left( \frac{\sqrt{6}}{\pi} \right) \)
\[\vdots\]
\( \frac{1}{n} \left( \frac{\sqrt{6}}{\pi} \right) \)
\( -\frac{2\sqrt{6}}{n\pi} \)
\[
\sum_{n} |c_n|^2 = \sum_{n \text{ odd}} |c_n|^2 = \\
= \left( \frac{\sqrt{6}}{\pi} \right)^2 \left( 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \cdots \right) \\
+ \left( \frac{2\sqrt{6}}{\pi} \right)^2 \left( \frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \cdots \right)
\]

\[
\left( \frac{1^2}{3^2} + \frac{1^2}{9^2} + \frac{1^2}{15^2} + \cdots \right) = \frac{1}{3^2} \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right)
\]

\[
= \frac{1}{3^2} \sum_{n \text{ odd}} \frac{1}{n^2} = \frac{1}{9} \left( \frac{\pi^2}{6} \right)
\]

\[
\left( 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \cdots \right) = \sum_{n \text{ odd}} \frac{1}{n^2} - \left( \frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \cdots \right)
\]

\[
= \sum_{n \text{ odd}} \frac{1}{n^2} - \frac{1}{9} \left( \sum_{n \text{ odd}} \frac{1}{n^2} \right) = \frac{8}{9} \left( \frac{\pi^2}{6} \right)
\]

\[
\sum_{n} |c_n|^2 = \left( \frac{\sqrt{6}}{\pi} \right)^2 \frac{8}{9} \left( \frac{\pi^2}{6} \right) + \left( \frac{2\sqrt{6}}{\pi} \right)^2 \frac{1}{9} \left( \frac{\pi^2}{6} \right)
\]

\[
= \frac{2}{3} + \frac{1}{3} = 1
\]
(4) \[ \langle \psi \mid H \mid \psi \rangle = \sum_n |c_n|^2 E_n \]

\[ E_n = \hbar^2 \left( \frac{k_n^2}{8mL^2} \right) \]

\[ |c_n|^2, \ n \ odd \sim \frac{1}{\hbar^2} \]

\[ \sum_n |c_n|^2 E_n \rightarrow \infty \]
Problem 4.

A long straight wire runs parallel to and a distance $h$ above the surface of a superconducting sheet. The wire carries current $I$, and magnetic field is excluded from the superconductor. Determine the force per unit length on the wire, and state explicitly whether the wire is attracted or repelled from the surface.

Hint: Use the method of images.
The b.c. is that $B_{n}=0$ at the surface of superconductor. Satisfied
for an image wire carrying current $-I$ a distance $h$ below the surface.

\[ (B_{0}) = \frac{2I}{C(2h)} \]

\[
\frac{\text{Force}}{\text{length}} \cdot C(B_{0})_{\text{image}} = \frac{2I^2}{C^2/2h}
\]

force is repulsive
Problem 5.

Use basic principles to derive order of magnitude estimates for the properties of the sun.

a) Obtain an order of magnitude estimate of the pressure in the core of the sun, $P_c$. Assume that the core has a radius $R_c$ that is 0.2 of the total radius $R_\odot = 7 \times 10^{10}$ cm and contains 0.3 of the total mass $m_\odot = 2 \times 10^{33}$ gm.

Hint: Find the "weight" of the mass contained in the region $R > R_c$; you may assume a constant gravitational field for $R > R_c$ for simplicity.

b) Obtain an order of magnitude estimate of the core temperature. Assume that the sun is all Hydrogen ($m = 1.5 \times 10^{-24}$ gm), and use the ideal gas law ($k = 1.4 \times 10^{-16}$ ergs K$^{-1}$).

c) For the average core pressure and temperature, the average opacity implies a mean free path of 3 mm for a photon. This is how far it travels before it interacts with matter. Calculate the order of magnitude time for a photon to random walk from the center to the surface.

d) Obtain an order of magnitude estimate of the luminosity of the sun, $L = \text{energy radiated per unit time}$. The energy density of thermal radiation is $E = aT^4$, $a = 7 \times 10^{-15}$ erg cm$^{-3}$ K$^{-1}$. Use earlier results.

$G = 6.67 \times 10^{-8}$ gm$^{-1}$ cm$^3$ s$^{-2}$
Hydrostatic Equilibrium

\[ P_c = \text{weight per unit area of gas on top} \]
\[ = (\text{mass per unit area}) \times \text{grav. field} \]
\[ = \frac{M}{4\pi (0.2R)^2} \times \frac{GM}{(0.2R)^2} \]
\[ = \frac{0.7M}{4\pi (0.2)^4 R^4} \times \frac{6.03M}{(0.2R)^2} \]
\[ = 0.7 \times 0.3 \times \frac{GM^2}{4\pi 0.2^4 R^4} \]
\[ = 10.4 \frac{GM^2}{R^4} \]
\[ = 10.4 \times 6.67 \times 10^{-8} \text{ gm}^{-1} \text{cm}^3 \text{gm}^{-2} \times (2 \times 10^{33} \text{ gm})^2 / (7 \times 10^{10} \text{ cm})^4 \]
\[ P_c = 1.2 \times 10^{17} \text{ gm cm}^{-1} \text{ s}^{-2} \]
b) \( P_c = n_c k T_c \)

\[ n_c = \frac{\rho_c}{m_H} = \left( \frac{M_c}{\frac{4\pi}{3} R_c^3} \right) m_H \]

\[ \rho_c = \frac{0.3 M}{\frac{4\pi}{3} (0.2 R)^3} = \frac{0.3 \times 2 \times 10^{33} \text{ gm}}{\frac{4\pi}{3} (0.2 \times 7 \times 10^6 \text{ cm})^3} = 52 \text{ gm}^{-3} \]

\[ n_c = \frac{\rho_c}{m_H} = \frac{52 \text{ gm}^{-3}}{1.5 \times 10^{-24} \text{ gm}} = 3.5 \times 10^{25} \text{ protons/cm}^3 \]

\[ T_c = \frac{P_c}{n_c k_B} = \frac{1.2 \times 10^{17} \text{ gm cm}^{-3} \text{ s}^{-1}}{3.5 \times 10^{25} \times 1.38 \times 10^{-16} \text{ erg K}^{-1}} \]

\[ T_c = 2.5 \times 10^7 \text{ K} \]
c) Distance travelled in random walk
\[ d = l \times \sqrt{N} = \text{mean free path} \times \sqrt{\text{number steps}} \]

Number steps to escape travel \( R \) is
\[ N = \left( \frac{R}{t} \right)^2 = \left( \frac{7 \times 10^{10} \text{cm}}{0.3 \text{ cm}} \right)^2 = 5 \times 10^{22} \text{ steps} \]

Time for photon to travel \( N \) steps
\[ t = (\text{time/step}) \times N \]
\[ t = \frac{l}{c} \times N = \frac{0.3 \text{ cm}}{3 \times 10^{10} \text{ cm/s}} \times 5 \times 10^{22} = 5 \times 10^{11} \text{ s} \]
d) \[ L = \text{Energy radiated/second} \]
\[ = \frac{\text{energy density} \times \text{volume}}{\text{time} \times \text{random walk R}} \]
\[ = \frac{8 \pi a T_c^4}{3} \times \frac{4 \pi}{3} R_c^3 \]
\[ \left( \text{from (c)} \right) \]
\[ \left( \text{from (b)} \right) \]
\[ = \frac{7 \times 10^{-15} \text{erg cm}^{-3} \text{K}^4}{5 \times 10^{11} \text{s}} \times (2.5 \times 10^7)^4 \times \frac{4 \pi}{3} (0.2 R)^3 \]
\[ \left( \text{from (c)} \right) \]
\[ = 6 \times 10^{34} \text{ erg s}^{-1} \]
Problem 6.

The conduction electrons in a metal have mean free time \( \tau = 2 \cdot 10^{-13} \) s and density \( n_0 = 10^{20} \) cm\(^{-3}\). An electromagnetic wave of frequency \( 1.1 \times 10^{14} \) Hz is normally incident on the metal.

a) Calculate the dielectric constant of conduction electrons and the reflection coefficient for a wave with frequency \( \omega \).

b) For the above listed parameters, what fraction of the incident power is reflected? 1% accuracy is sufficient

Hint: Use for reflection coefficient at normal incidence the formula \( R = \frac{|n-1|^2}{|n+1|} \), where \( n(\omega) \) is the refraction index, and assume that \( \mu = \mu_0 \) is a good approximation, where \( \mu_0 \) is the magnetic permeability of vacuum.
Solution

Equation of motion for conductivity electrons in $\vec{E}$, $\vec{B}$ fields of electromagnetic wave:

$$m_e \frac{d\vec{v}_e}{dt} = -e \vec{E} - \frac{e}{c} \vec{v}_e \times \vec{B} - me \frac{1}{\epsilon} \vec{v}_e$$

Neglecting Lorentz force term quadratic in wave amplitude and taking into account that $\vec{E}(t) \sim \exp(-i\omega t)$, we have

$$\vec{v}_e = \frac{e \vec{E}}{me (i\omega - \frac{1}{c})}$$

and hence induced current

$$\vec{J}_e = -n_e e \vec{v}_e = - \frac{n_e e^2}{i me} \vec{E} \frac{1}{\omega + \frac{i}{c}} = \tilde{\sigma} \vec{E}$$

Hence H.F. conductivity

$$\tilde{\sigma} = \frac{i n_e e^2}{l}$$
where

\[ \omega_p^2 = \frac{4\pi e^2 n_0}{m_e} \] - square of plasma frequency for electrons in a conductivity zone.

For \( n_0 = 10^{20} \text{ cm}^{-3} \), \( \omega_p = 5.48 \times 10^6 \).

For \( \mu = 1 \) refractive index \( n(\omega) = \sqrt{\varepsilon} \) and reflection coefficient in the case of normal incidence:

\[
R = \left| \frac{\sqrt{1 - \frac{\omega_p^2}{\omega^2}} - \frac{1}{1 + \frac{i}{\omega c}}}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}} + \frac{1}{1 + \frac{i}{\omega c}}} \right|^2
\]
\[ R = \left| \frac{\sqrt{0.45 + i \cdot 0.007} - 1.1}{\sqrt{0.45 + i \cdot 0.007} + 1.1} \right|^2 = \]

\[ = \left| \frac{0.43 - i \cdot 0.005}{1.77 + i \cdot 0.005} \right|^2 \approx 6 \cdot 10^{-2} \]
Physics Departmental Examination - Spring 1997 - Part 1

Student Identification #

Problem 7.

Consider a chain with \( N \gg 1 \) massless links of length \( \ell \) that can be oriented with equal probability in three directions with respect to the link above: left, right, or down, as shown below. Suppose that the upper end of the chain is fixed, a constant force \( F \) is applied to the lower end of the chain, and the system is in thermodynamic equilibrium at temperature \( T \).

a) What is the mean vertical extension, \( \bar{L}_z \), of the chain?

b) What is the mean internal energy, \( \bar{E} \), of the chain?

c) Consider the high and low temperature limits, \( F\ell \ll k_B T \) and \( F\ell \gg k_B T \), respectively. Solve for \( \bar{L}_z \), \( \bar{E} \), and estimate the entropy \( S \) of the chain in these limits.

d) Estimate the expected fluctuation \( \Delta L_z \) in the mean vertical extension, \( \bar{L}_z \), in the high temperature limit, \( F\ell \ll k_B T \). [In this part, an answer good to a factor of 2 will be fine.]
Orienting a link downward does work \( W = Fl \) on it.

Thus \( E_d = -Fl \), \( \epsilon_{d, e} = 0 \)

\[
P_{r, e} = \frac{1}{2 + e^{-\beta Fl}}
\]

\[
P_{d} = \frac{e^{\beta Fl}}{2 + e^{\beta Fl}}
\]

(a) \[
\bar{L} = \frac{N \bar{L}}{2e^{-\beta Fl} + 1}
\]

(b) \[
\bar{E} = \frac{-N Fl}{2 + e^{\beta Fl}} = \frac{-N Fl}{2e^{-\beta Fl} + 1} \quad \{ \equiv -Fl \}
\]

(c) At high \( T \), \( P_{d} = \theta_{d} = \theta_{L} \)

\[
\bar{L} = \frac{N \bar{L}}{3} \quad \bar{E} = -\frac{N Fl}{3}
\]

\[
S = k_{B} \ln(2) = k_{B} \ln(3) = \frac{N}{k_{B}} \ln 3
\]
(c) cont'd

At low $T$ $P_d = 1$ $P_x = P_e = 0$

$L = N L$ $E = -NFE$

$S = k_B \ln N' = 0$ → (gauge)

(d) $\Delta N_d \approx \sqrt{N_d}$ At high $T$ $P_d = 0 = P_e = \frac{1}{3}$

$$\frac{\Delta L}{L} = \frac{\Delta N_d}{N_d} = \frac{1}{\sqrt{N_d}}$$

$$\Delta L' \approx \sqrt{N_d} \frac{L}{\sqrt{N_d}} = \sqrt{\frac{N'}{3}} L$$

Really $\Delta N' = \sqrt{N \left( \frac{1}{3} \right) \left( \frac{2}{3} \right)}$

$$\Delta L' \approx \sqrt{\frac{2N}{9}} L$$
Physics Departmental Examination - Spring 1997 - Part 1

Problem 8.

The shift of the ground state energy of a hydrogen atom in the presence of a constant, uniform electric field $E$, is given by (for small $E$)

$$\Delta E = \frac{1}{2} \alpha E^2$$

where $\alpha$ is the polarizability.

a) Find an expression for $\alpha$ using perturbation theory.

b) From your result in (a), find an upper bound for $\alpha$ of the form

$$\alpha_{\text{max}} = c a_0^3$$

with $a_0 = \frac{\hbar^2}{m_e e^2}$ the Bohr radius and $c$ a constant. Find the numerical value of $c$.

Hint: Use the relation

$$\sum_m |A_{nm}|^2 = (A^2)_{nn}$$

where $A$ is a quantum-mechanical operator, $m$ a complete set of states, and $A_{nm}$ the matrix element of $A$ between states $n$ and $m$. 
Solution: Spring 1997
Prob. 8

The nuclear potential is

\[ V = e^2/r \]

\[ n = \frac{1}{\sqrt{n^3}} \]

The ground state is

\[ g_0 = \frac{1}{\sqrt{n^2}} \]

The ground energy shift is

\[ \Delta E^{(1)} = \langle g_0, V g_0 \rangle = 0 \]

Since \[ \int 1_{r} e^{-2\alpha r} = 0 \]

The second energy shift is

\[ \Delta E^{(2)} = \sum_{n>0} \frac{1}{E_n - E_0} \left| \langle g_0, z g_n \rangle \right|^2 = e^2 \sum_{n>0} \frac{1}{E_n - E_0} \left| \langle g_0, z g_n \rangle \right|^2 \]

\[ E - E_0 \]

\[ \delta = \frac{e}{E - E_0} \sum_{n>0} \frac{\left| \langle g_0, z g_n \rangle \right|^2}{E_n - E_0} \]

\[ \delta \leq \frac{e}{E_1 - E_0} \sum_{n>1} \frac{\left| \langle g_0, z g_n \rangle \right|^2}{E_n - E_0} \]

\[ E_n \leq 1 \]

\[ \frac{1}{E_n - E_0} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ n+1 \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]

\[ E_{n+1} \]

\[ E_1 \]

\[ E_0 \]

\[ E_n \]
\[ S_0, \mathbb{Z} \mathbb{Z} J_0 = \frac{1}{\pi a_0^3} \int d^3 r \, r^2 e^{-2r/a_0} = \frac{1}{3 \pi a_0^3} \int d^3 r \, r^4 e^{-2r/a_0} = \frac{4}{3a_0^3} \int d r \, r^4 e^{-2r/a_0} = \]

\[ = \frac{a_0^2}{12} \int_0^\infty dx \, x^4 e^{-x} = \frac{4! \, a_0^2}{12} = 2a_0^2 \]

\[ \chi_{\text{m.o.}} = \frac{2e^2 a_0^2}{E_1 - E_0} \]

\[ E_0 = -\frac{e^2}{2a_0}, \quad E_1 = \frac{E_0}{4} \Rightarrow E_1 - E_0 = -\frac{3}{4} E_0 = +\frac{3}{4} \frac{e^2}{2a_0} \]

\[ \chi_{\text{m.o.}} = \frac{2e^2 a_0^2 \cdot 4 \cdot 2a_0}{3 e^2} = \]

\[ \chi_{\text{m.o.}} = \frac{16 a_0^3}{3} \]

\[ C = \frac{16}{3} \]
Problem 9.

Two infinite conducting sheets are placed so that one is in the half plane \( y = 0, x > 0 \) and the other is in the half plane \( y = x, x > 0 \). The two sheets are electrically insulated from each other. The sheet at \( y = 0 \) is kept at zero potential, the sheet at \( y = x \) is kept at a potential \( V_0 \).

a) Solve for the potential between sheets.

b) Find electric field components in cartesian coordinates.

Hint: Laplacian in cylindrical coordinates is

\[
\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2}
\]
Solution.

Since the potential is homogeneous in $z$ direction, the Laplace equation is of the form

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \varphi^2} = 0$$

Separating variables, we have

$$V(r, \varphi) = R(r) \phi(\varphi)$$

where

$$\phi'' + m^2 \phi = 0$$

$$\phi = A \sin m \varphi + B \cos m \varphi, \quad m \neq 0$$

$$\phi = A \varphi + B \quad \text{if} \quad m = 0.$$  

Boundary condition for $\varphi = 0$ gives

$$B = 0$$

For $R(r)$ we have an equation

$$R'' + \frac{1}{r} R' - \frac{m^2}{r^2} R = 0$$
\[ R(r) = A_1 r^m + B_1 r^{-m} \quad \text{if } m \neq 0 \]

\[ R(r) = A_1 \ln r + B_1 \quad \text{if } m = 0 \]

In both conducting sheets tangential electric field is equal to zero. Hence \( R = \text{const} \Rightarrow m = 0 \) and \( A_1 = 0 \)

Solution satisfying boundary condition at \( y = x \) plane has a form:

\[ V = \frac{4V_0}{\pi} \varphi \]

\[ E_\varphi = -\frac{1}{r} \frac{\partial V}{\partial \varphi} = -\frac{4V_0}{\pi r} ; \quad E_x = 0 \]

\[ E_x = -E_\varphi \sin \varphi = -\frac{4V_0 y}{\pi (x^2 + y^2)} ; \quad E_y = E_\varphi \cos \varphi = -\frac{4V_0 x}{\pi (x^2 + y^2)} . \]
Problem 10.

A charged particle, mass $m$ and charge $e$, is constrained to move in the plane around a fixed charge $-e$. In addition, there is a uniform magnetic field applied perpendicular to the plane. The particle is initially at rest a distance $r_0$ from the fixed charge. Find its distance of closest approach to the fixed charge during the subsequent motion.

(Assume $\frac{B^2 r_0^3}{mc^2} \ll 1$).

Hints: (i) the vector potential in cylindrical coordinates is $\vec{A} = \frac{Br}{2} \hat{\theta}$

(ii) use conservation laws
Solution: Spring 1997
Prob. 10

\[ H = \frac{1}{2} m v^2 - \frac{e^2}{r} = -\frac{e^2}{c_0} \text{ initially} \]

\[ L = \frac{m v_0 c + e B r^2}{2 c} = \frac{e B r_0^2}{2 c} \text{ initially} \]

min. approach distance: \( v_0 = 0 \)

\[ \Rightarrow H = \frac{1}{2} m v_0^2 - \frac{e^2}{r_{\text{min}}} = -\frac{e^2}{c_0}, \quad r_{\text{min}} = \frac{c_0}{2} \]

\[ m v_0 c_{\text{min}} = e B (r_0^2 - r_{\text{min}}) \Rightarrow v_0 = \frac{e B (r_0^2 - r_{\text{min}})}{2 c r_{\text{min}}} \]

\[ c_{\text{min}}^2 = \frac{e^2}{2 m} \left( \frac{r_0^2}{2 r_{\text{min}}} \right)^2 + \frac{e^2}{r_{\text{min}}} \]

\[ \left( \frac{c_{\text{min}}}{r_0} \right)^2 = \frac{B r_0}{\gamma m c^2} \left[ (1 - \frac{c_{\text{min}}}{r_0}) \right] - \frac{c_{\text{min}}}{r_0} \]
\[ x = \frac{\alpha}{B_c^3} \]

\[ -x^2 = \alpha (1 - 2x^2 + x^4) \quad \text{x quartic, can't be solved} \]

but assume that \[ \frac{B_c^2}{mc^2} << 1 \Rightarrow x << 1 \] is what we want

asymptotic solution: \[ x = \alpha \]

powers dominate below

for \( \alpha << 1 \), \( x << 1 \)
Problem 11.

A gas satisfying the relation

$$E(S, V, N) = aS^4 / NV^2, \quad a = \text{constant}$$

is run through the following thermodynamic cycle:

- AB: isotherm (dT = 0)
- AC: adiabat (dQ = 0)
- BC: isochole (dV = 0)

The volume ratio is $\frac{V_C}{V_A} = 2$. The particle number N is constant.

(a) Compute $\frac{P_B}{P_A}$

(b) Compute $\frac{P_C}{P_A}$

(c) Compute $\frac{T_C}{T_A}$
Solution: Problem II

Given \( E(S,V,N) = aS^4/VN^2 \) we have

\[
T = \frac{\partial E}{\partial S}_{S,N} = \frac{4aS^3}{NV^2} \quad \text{(0)}
\]
\[
p = -\frac{\partial E}{\partial V}_{S,N} = 2aS^4/NV^3 \quad \text{(0)}
\]

Eliminating \( S \),

\[
T^4 = 2^{\frac{8}{3}}a^4S^{12}/N^4V^8 \quad \{ \Rightarrow \quad 32aVp^3 = NT^4 \quad \text{(3)} \}
\]
\[
p^3 = 2^{\frac{3}{2}}a^3S^{12}/N^3V^9 \quad \}

Thus, with \( dN = 0 \),

isotherm \( \Rightarrow Vp^3 = \text{constant} \) (from (3))

adiabat \( \Rightarrow pV^3 = \text{constant} \), \( TV^2 = \text{constant} \) (from (0),(0))

(a) \( AB \) isotherm \( \Rightarrow \)

\[
V_A p_A^3 = V_B p_B^3
\]
\[
\Rightarrow \quad \frac{p_B}{p_A} = \left( \frac{V_A}{V_B} \right)^{\frac{4}{3}} = 2^{-\frac{4}{3}}
\]

(b) \( AC \) adiabat \( \Rightarrow \)

\[
p_A V_A^3 = p_C V_C^3 \quad \Rightarrow \quad \frac{p_C}{p_A} = \left( \frac{V_A}{V_C} \right)^3 = \left( \frac{V_A}{V_B} \right)^3 = \frac{1}{8}
\]

(c) \( AC \) adiabat \( \Rightarrow \)

\[
T_A V_A^2 = T_C V_C^2 \quad \Rightarrow \quad \frac{T_C}{T_A} = \left( \frac{V_A}{V_C} \right)^2 = \left( \frac{V_A}{V_B} \right)^2 = \frac{1}{4}
\]
Solution: Problem 1.

Given:
\[ E(s, v, N) = \frac{a s^4}{N V^2} \]

we have:
\[ T = \left( \frac{\partial E}{\partial s} \right)_{v, N} = \frac{4a s^3}{N V^2}, \quad V \]
\[ p = -\left( \frac{\partial E}{\partial N} \right)_{s, v} = \frac{2a s^4}{N V^3} \]

Eliminating \( s, \)
\[ T^4 = 2^8 a^4 s^4 / N^4 V^8 \]
\[ p^3 = 2^3 a^3 s^4 / N^3 V^3 \]
\[ \Rightarrow \quad 32a V p^3 = N T^4 \]

Thus, with \( dN = 0, \)

isotherm \( \Rightarrow \) \( V p^3 = \) constant (from \( a \))

adiabat \( \Rightarrow \) \( p V^3 = \) constant \( , \) \( TV^2 = \) constant (from \( a, \( b \))

(a) \( AB \) isotherm \( \Rightarrow \)
\[ V_A p_A^3 = V_B p_B^3 \]
\[ \Rightarrow \quad \frac{p_B^3}{p_A^3} = \left( \frac{V_A}{V_B} \right)^4 = 2^{-2/3} \]

(b) \( AC \) adiabat \( \Rightarrow \)
\[ p_A v_A^3 = p_C v_C^3 \]
\[ \Rightarrow \quad \frac{p_C}{p_A} = \left( \frac{V_A}{V_C} \right)^3 = \left( \frac{V_A}{V_B} \right)^3 = \frac{1}{8} \]

(c) \( AC \) adiabat \( \Rightarrow \)
\[ T_A v_A^2 = T_C v_C^2 \]
\[ \Rightarrow \quad \frac{T_C}{T_A} = \left( \frac{V_C}{V_A} \right)^2 = \frac{V_A}{V_B} = \frac{1}{9} \]
Solution: Spring 1997
Prob. 12

Schrodinger eq. 15

\[
\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) \right] \Psi(r) = E \Psi(r)
\]

In the absence of potential, eigenfunctions are plane waves:

\[\Psi_n(r) = \frac{1}{\sqrt{V}} e^{i \mathbf{k} \cdot \mathbf{r}} \quad \text{with energy} \quad E_n = \frac{\hbar^2 k^2}{2m}\]

Expand solution in plane waves:

\[\Psi(r) = \sum \alpha_n \Psi_n(r)\]

\[
\frac{-\hbar^2}{2m} \nabla^2 + V(r) \sum \alpha_n \Psi_n(r) = E \sum \alpha_n \Psi_n(r) = \Rightarrow
\]

\[
\sum \alpha_n [E_n + V(r)] \Psi_n(r) = E \sum \alpha_n \Psi_n(r)
\]

Apply by \(\Psi_n^*(r)\) and integrate over \(r\):

\[
\int d^d r \Psi_n^*(r) \Psi_n(r) V(r) = \delta_{nn'} \Rightarrow
\]

\[
E_n \Psi_n + \sum \alpha_n \int d^d r \Psi_n^*(r) \Psi_n(r) V(r) = E \alpha_n
\]

In uniform potential:

\[
\int d^d r \Psi_n^*(r) \Psi_n(r) V(r) = \frac{-\hbar^2}{V} \int d^d r e^{i \mathbf{k} \cdot \mathbf{r}} = \frac{-\hbar^2}{V} \]

\[
(E-E_n) \alpha_n = -\frac{\hbar^2}{V} \sum \alpha_n \rightarrow \alpha_n = -\frac{\hbar^2}{V} \frac{1}{E-E_n} \sum \alpha_n
\]

Summing both sides over \(n\):

\[
\sum \alpha_n = -\frac{\hbar^2}{V} \left[ \frac{1}{E-E_n} \right] \sum \alpha_n \quad \text{can dim. w/} \sum \alpha_n = 0
\]

\[
1 = -\frac{\hbar^2}{V} \sum \frac{1}{E-E_n}
\]

Transforming to an integral:\n
\[
\sum \rightarrow \frac{V}{(2\pi)^d} \quad V = \text{volume in d dimensions}
\]

\[
1 = -\frac{\hbar^2}{(2\pi)^d} \int d^d k \frac{1}{E-E_n}
\]
1. is a bound state, \( E < 0 \). For small \( \mu \), \( E \) will be close to 0. For infinite \( \mu \), there will be a bound state if the integral diverges as \( E \to 0^- \):

\[
\sum_{k=0}^{\infty} \int_0^1 \frac{d^k \hbar}{E_{k} - E} \to \sum_{k=0}^{\infty} \frac{d^k \hbar}{\hbar^2} \sim \sum_{k=0}^{\infty} \frac{\hbar^d}{\hbar^2} = 1,
\]

\[
\int_0^{\infty} \frac{d\hbar}{\hbar^2}
\]

diverges \( \Rightarrow \) there is a bound state for infinite \( \mu \).

2. \( \int_0^{\infty} \frac{d\hbar}{k} \) which diverges \( \Rightarrow \) there is a bound state for infinite \( \mu \).

3. \( \int_0^{\infty} d\hbar \) does not diverge \( \Rightarrow \) a finite \( \mu \) is required to get a bound state.

Minimum \( \mu \) required in \( d = 3 \):

\[
\frac{\mu}{(2\pi)^2} \int \frac{d^2 \hbar}{E_{k}} = \frac{\mu}{(2\pi)^2} \frac{4\pi}{k^2} \int_0^{\infty} d\hbar = \frac{\mu m}{\pi^2 k^2}.
\]

The minimum \( \mu \) required is

\[
\mu = \frac{\pi^2 k^2}{m \hbar}.
\]
Problem 13.

A particle moves in one dimension with Hamiltonian

$$H = \varepsilon(p) - Fx$$

where $\varepsilon(p)$ is periodic with period $P$.

1. What are the equations of motion?

2. Show that the coordinate transformation

$$x' = x$$
$$p' = p - Ft$$

is canonical and find the generator.

3. Find the Hamiltonian $H'(x', p', t)$.

4. What are the equations of motion for $x'$ and $p'$? Use these equations to demonstrate that the motion is periodic. What is the period? (Remark: This part of the problem can be done even if you could not do parts 2 and 3).
Problem A particle moves in one dimension with Hamiltonian

\[ \mathcal{H} = \mathcal{E}(p) - Fz, \]

where \( \mathcal{E}(p) \) is periodic with period \( P \).

(1) What are the equations of motion?

(2) Show that the coordinate transformation

\[
\begin{align*}
  z' &= z \\
  p' &= p - Ft
\end{align*}
\]

is canonical and find the generator.

(3) Find the Hamiltonian \( \mathcal{H}'(x', p', t) \).

(4) What are the equations of motion for \( z' \) and \( p' \). Use these equations to demonstrate that the motion is periodic. What is the period? (Remark: This part of the problem can be done even if you could not do parts 2 and 3.)

Solution

(1) The equations of motion are

\[
\begin{align*}
  \dot{z} &= \mathcal{E}'(p) \\
  \dot{p} &= F
\end{align*}
\]

(2) The Poisson brackets are obviously correct, so the new coordinates are canonical. To find the generator, write

\[ p' dx' - \mathcal{H}' dt = pdx - \mathcal{H} dt + df. \]

We have \( \mathcal{H} = -Ftx \) and

(3) \[ \mathcal{H}' = \mathcal{H} + Fz = \mathcal{E}(p) = \mathcal{E}(p' + Ft). \]

(4) Writing Hamilton's equations using \( \mathcal{H}' \) or by direct transformation of the equations of motion for \( z \) and \( p \), we have

\[
\begin{align*}
  \dot{z}' &= \mathcal{E}'(p' + Ft) \\
  \dot{p}' &= 0.
\end{align*}
\]

So \( p' \) is constant and \( z' \) is obtained by integration. Because \( \mathcal{E} \) is periodic, the \( z' \) motion is periodic with period \( P/F \). (Note that the average of \( \dot{z}' \) is zero because of the periodicity of \( \mathcal{E} \).)
Problem 14.

An electromagnetic wave of frequency $\omega$ propagates along the x axis in the $x = -\infty$ direction. It collides with a slab of dielectric moving with velocity $v$ in the positive x direction. There are both reflected and transmitted waves. Find the frequencies of these waves if the motion of the dielectric is relativistic, i.e. $v/c \sim 1$. 
Solution:

The reference frame of the dielectric of both reflected and transmitted waves have the same frequency \( \omega_0 \). To find \( \omega_0 \) we use Lorentz transformation:

\[
x' = \frac{x + V^2 t}{\sqrt{1 - V^2/c^2}}, \quad t = \frac{t' + \frac{V}{c^2} x'}{\sqrt{1 - V^2/c^2}}.
\]

The prime refers to a frame moving with a velocity \( V \).

A plane wave \( \exp[-i \omega_0 t' + i \frac{\omega}{c} \cdot \hat{n} \cdot \vec{x}'] \) is a unit vector in the direction of wave propagation is transformed into:

\[
\exp[-i \omega \left( \frac{1}{\sqrt{1 - V^2/c^2}} - \frac{V_n}{c} \frac{1}{\sqrt{1 - V^2/c^2}} \right) t']
\]

\[+ i \frac{\omega}{c} \left( \frac{1}{\sqrt{1 - V^2/c^2}} - \frac{V_n}{c} \frac{1}{\sqrt{1 - V^2/c^2}} \right) (x' \hat{n})]
\]

\[
\exp \left[ -i \omega_0 t' + i \frac{\omega_0}{c} (x' \hat{n}) \right]
\]

and

\[
\omega_0 = \frac{\omega (1 - V_n/c)}{\sqrt{1 - V^2/c^2}}.
\]
where $V_n$ is a projection of $V$ in the direction of wave propagation. For incident wave $V_n = -V$

$$
\omega_0 = \omega \frac{1 + \frac{V}{c}}{\sqrt{1 - \frac{V^2}{c^2}}} = \omega \frac{1 + \frac{V}{c}}{\sqrt{1 - \frac{V^2}{c^2}}}
$$

Using the same relationship for both reflected ($V_n = V$) and transmitted ($V_n = -V$) waves we have

$$
\omega_0 = \omega_t \frac{1 - \frac{V}{c}}{\sqrt{1 - \frac{V^2}{c^2}}} = \omega \frac{1 + \frac{V}{c}}{\sqrt{1 - \frac{V^2}{c^2}}}
$$

$$
\omega_0 = \omega_t \frac{1 + \frac{V}{c}}{\sqrt{1 - \frac{V^2}{c^2}}} = \omega \frac{1 + \frac{V}{c}}{\sqrt{1 - \frac{V^2}{c^2}}}
$$

Hence for a transmitted wave

$$\omega_t = \omega$$

and for a reflected wave

$$\omega_r = \omega \frac{1 + \frac{V}{c}}{1 - \frac{V}{c}}$$

Well known effect of frequency multiplication as the result of reflection.
Physics Departmental Examination - Spring 1997 - Part 2

Student Identification # ____________

Problem 15.

Consider N classical spins of \( s = \{ \pm 1 \} \) on a one-dimensional lattice. The interaction of spins on site \( m \) and \( n \) (denoted by \( s_m \) and \( s_n \) respectively) is given by the exchange term \( J_{mn} \), so that the total energy of a given configuration of spins \( \{ s_i \} \) is

\[
H = - \sum_{m,n=1}^{N} J_{mn} s_m s_n
\]

(a) Show that if every spin is weakly coupled to each other, i.e., \( J_{mn} = J/N \), then spontaneous magnetization occurs below a critical temperature \( T_c \) in the thermodynamic limit \( N \to \infty \). Find the critical temperature as a function of \( J \).

(b) Show that \( T_c = 0 \) for nearest-neighbor coupling, \( J_{mn} = J \delta_{m,n+1} \), i.e., spontaneous magnetization cannot exist in the thermodynamic limit at any finite temperature \( T > 0 \). (Note: You do not need to solve the full problem to reach this conclusion).

Hint: Consider the energy and entropy of low energy excitations above the ground state and how they contribute to the free energy.

(c) Suppose the coupling has the form \( J_{mn} = J/|m - n|^{\sigma} \) where \( \sigma \) is a positive real number. Extending the argument of (b), show that \( T_c > 0 \) can result only if \( \sigma \) is smaller than a threshold value \( \sigma_c \) and find \( \sigma_c \).
Solution:

(a) When the spins are coupled to each other, we have the mean-field situation. Let the average magnetization be $\bar{s} = N^{-1} \sum_{m=1}^{N} s_m$. The total energy becomes

$$\mathcal{H} = -J \sum_{n=1}^{N} s_n \bar{s}.$$  

The thermally average of magnetization per site, $\langle s \rangle$, is given by the Boltzmann weight $e^{-\mathcal{H}/k_B T}$ as

$$\langle s \rangle = \sum_{s=\pm 1} s e^{-J s \bar{s}/k_B T} \Big/ \sum_{s=\pm 1} e^{-J s \bar{s}/k_B T} = \tanh (J \bar{s}/k_B T).$$

In the thermodynamic limit, spatial average and ensemble average are equivalent, hence, we have the self-consistency condition

$$\bar{s} = \tanh (J \bar{s}/k_B T).$$  \hspace{1cm} (1)
Spontaneous magnetization occurs when Eq. (1) admits (stable) solution at non-zero value of $s$. This occurs at $T_c = J/k_B$.

(b) The nearest-neighbor Ising model can be solved exactly using the transfer matrix method. However, to show the absence of spontaneous magnetization, the following consideration suffices.

- At $T = 0$, the ground state is clearly one with all spins taking on the same value (+1 or -1).
- Low energy excitations are "kink" configurations where a contiguous segment of the spins take on one value and the rest of the spins take on the opposite value, say, $s_m = +1$ for $1 < m < m^*$, and $s_m = -1$ for $m^* < m < N$. The energy cost of such an excitation is $\Delta E_{\text{kink}} = +2J$ for all kink position $m^*$ as long as it is not at the two ends of the 1d lattice.
- Since the kink configuration has a degeneracy of the order $N$, the entropy gain is $\Delta S = \log N$. The total free energy cost of the kink configuration is

$$\Delta F_{\text{kink}} = 2J - k_B T \log N$$

which is negative at any finite temperature $T > 0$ for a sufficiently large $N$. Hence the ground states with uniform magnetization are unstable, replaced by kink configurations which have zero average magnetization.

(c) Continuing along the line of consideration given in (b), we need to estimate the energy cost of a kink configuration for the non-local coupling $J_{mn}$—

$$\Delta E_{\text{kink}} = 2J \sum_{n=m^*+1}^{N} \sum_{m=1}^{m^*-1} \frac{1}{|m-n|^\sigma}.$$  \hspace{1cm} (2)

For kink position not too close to the ends of the system, say $N/4 \lesssim m^* \lesssim 3N/4$, and for $N \gg 1$, the sums in Eq. (2) may be approximated by integrals, yielding the result

$$\Delta E_{\text{kink}} \sim O(J N^{2-\sigma})$$

to the leading order in $N$. This energy is larger than the entropy gain of kink formation, still of the order $\log N$, as long as $\sigma < 2$. Hence the ground state is stable and spontaneous magnetization occurs provided $\sigma < \sigma_c = 2$. 

\[ \begin{array}{ccccccccc} + & + & + & \underbrace{---} & \cdots \end{array} \]

2
Problem 16.

Find the energy spectrum of a particle of charge $q$ and mass $m$ moving in crossed homogeneous electric and magnetic fields $\vec{E} = E\hat{x}$, $\vec{B} = B\hat{z}$.

Hints:
1) Choose gauge where the magnetic vector potential points in the $y$ direction
2) Transform in the Hamiltonian from the variable $x$ to a new variable

$$X = x - \frac{c}{qB} p_y - \frac{mc^2E}{qB^2}$$

where $p_y$ is momentum in $y$ direction and $c$ is the velocity of light.
$A_y = B x$, then $\nabla \times \vec{A} = \vec{B}$

\[ H = \frac{p_x^2}{2m} + \frac{(p_y - \frac{q}{c} B x)^2}{2m} + \frac{p_z^2}{2m} - q E \cdot \hat{x} \]

Transform:

\[ X = x - \frac{c}{QB} p_y - \frac{mc^2 E}{QB^2} \Rightarrow \frac{c}{QB} (\frac{qB}{c} x - p_y) - \frac{mc^2 E}{QB^2} \]

\[ \Rightarrow X^2 = \frac{c^2}{q^2 B^2} (p_y - \frac{qB}{c} x)^2 - 2 \frac{mc^2 E}{QB^2} (x - \frac{c}{QB} p_y) + \frac{m^2 c^4 E^2}{q^2 B^4} \Rightarrow \]

\[ (p_y - \frac{qB}{c} x)^2 = \frac{q^2 B^2}{2mc^2} X^2 + \frac{c^2 E qB}{QB^2} (x - \frac{c}{QB} p_y) - \frac{mc^2 E^2}{QB^2} \cdot \frac{qB}{c} \]

\[ = \frac{q^2 B^2}{2mc^2} X^2 + q E x - \frac{E c}{B} p_y - \frac{mc^2 E^2}{c^2 B^2} \]

\[ H = \frac{p_x^2}{2m} + \frac{q^2 B^2}{2mc^2} X^2 - \frac{E c}{B} \frac{p_y - mc^2 E^2}{c^2 B^2} + \frac{p_z^2}{2m} \]

\[ H_1 \text{ is Hamiltonian of harmonic oscillator}, \ [p_x, X] = \frac{h}{i}, \text{ and} \]

\[ \frac{1}{2} m \omega^2 = \frac{q^2 B^2}{2mc^2} \Rightarrow \omega = \frac{qB}{mc}, \text{ eigenvalues are } \varepsilon_n = \hbar \omega (n + \frac{1}{2}) \]

Now, $p_y$ and $p_z$ commute with $H_1$, so second part can be diagonalized separately.

Energy levels are

\[ E(n, p_y, p_z) = \hbar \omega (n + \frac{1}{2}) - \frac{E c}{B} p_y + \frac{p_z^2}{2m} - \frac{mc^2 E^2}{2B^2} \]